LSE estimation for 2 sinusoids with 1 at a fixed frequency

This document addresses the estimation of two concurrent sinusoids with very close frequencies using a fixed window size, with the frequency of one fixed at a given value. The estimation follows the frequency-domain LSE scheme, i.e. maximizing the projection of signal spectrum X onto the subspace spanned from the two windowed sinusoid spectra, i.e. W_1 and $W_2(f)$, where f is the unknown frequency.

Consider the following decompositions of a signal X:

$$\mathbf{X} = \lambda \mathbf{W}(f) + \mathbf{r} \tag{1}$$

$$\mathbf{X} = \lambda_1 \mathbf{W}_1 + \lambda_2 \mathbf{W}_2(f) + \mathbf{r} \tag{2}$$

The LSE solution of (1) is given by orthogonal project as

$$\lambda(f) = \frac{<\mathbf{X}, \mathbf{W}(f)>}{\|\mathbf{W}(f)\|^2}, \; \|\mathbf{r}\|^2 = \|\mathbf{X}\|^2 - |\lambda(f)|^2 \|\mathbf{W}(f)\|^2 \tag{3}$$

According to (3), the LSE estimate is obtained by finding frequency f that maximizes $|\lambda(f)|^2 \|\mathbf{W}(f)\|^2$. In many cases $\|\mathbf{W}(f)\|^2$ can be regarded as constant so that only $|\lambda(f)|^2$ needs to be maximized.

Generally speaking in (2) W_1 and W_2 are independent vectors and therefore not orthogonal to each other. To find a LSE solution we need the following orthogonalization:

$$\mathbf{V}_2 = \mathbf{W}_2 - \frac{<\mathbf{W}_2, \mathbf{W}_1>}{\|\mathbf{W}_1\|^2} \mathbf{W}_1 \tag{4}$$

after which the LSE problem is solved using the following orthogonal projections:

$$\mu_1 = \frac{<\mathbf{X}, \mathbf{W}_1>}{\|\mathbf{W}_1\|^2}, \; \mu_2 = \frac{<\mathbf{X}, \mathbf{V}_2>}{\|\mathbf{V}_2\|^2}, \; \|\mathbf{r}\|^2 = \|\mathbf{X}\|^2 - |\mu_1|^2 \|\mathbf{W}_1\|^2 - |\mu_2(f)|^2 \|\mathbf{V}_2(f)\|^2 . \tag{5}$$

The decomposition is given by

$$\mathbf{X} - \mathbf{r} = \mu_1 \mathbf{W}_1 + \mu_2 \mathbf{V}_2 = \left(\mu_1 - \mu_2 \frac{<\mathbf{W}_2, \mathbf{W}_1>}{\|\mathbf{W}_1\|^2} \right) \mathbf{W}_1 + \mu_2 \mathbf{W}_2 \tag{6}$$

Since W_1 is fixed, the LSE estimate is obtained by finding frequency f that maximizes $|\mu_2(f)|^2 \|\mathbf{V}_2(f)\|^2$. Unlike W_1 or W_2, the norm of V_2 heavily depends on f, so $|\mu_2(f)|^2 \|\mathbf{V}_2(f)\|^2$ must be maximized as a whole. We write

$$<\mathbf{X}, \mathbf{V}_2> = <\mathbf{X}, \mathbf{W}_2 - \frac{<\mathbf{W}_2, \mathbf{W}_1>}{\|\mathbf{W}_1\|^2} \mathbf{W}_1> = <\mathbf{X} - \frac{<\mathbf{X}, \mathbf{W}_1>}{\|\mathbf{W}_1\|^2} \mathbf{W}_1, \mathbf{W}_2> \equiv <\mathbf{r}_1, \mathbf{W}_2> \tag{7}$$

$$\|\mathbf{V}_2\|^2 = \|\mathbf{W}_2\|^2 - \frac{<\mathbf{W}_2, \mathbf{W}_1><\mathbf{W}_2, \mathbf{W}_1>^*}{\|\mathbf{W}_1\|^2} \tag{8}$$

$$|\mu_2|^2 \|\mathbf{V}_2\|^2 = \frac{<\mathbf{X}, \mathbf{V}_2><\mathbf{X}, \mathbf{V}_2>^*}{\|\mathbf{V}_2\|^2} = \frac{<\mathbf{r}_1, \mathbf{W}_2><\mathbf{r}_1, \mathbf{W}_2>^*}{\|\mathbf{V}_2\|^2} = \frac{u}{v} \equiv y \tag{9}$$

To maximize y regarding f (which affects y through $\mathbf{W}_2(f)$), we need to calculates its 1st- and 2nd- order derivatives. That is

$$\frac{\partial y}{\partial f} = \frac{v \frac{\partial u}{\partial f} - u \frac{\partial v}{\partial f}}{v^2}, \; \frac{\partial^2 y}{\partial f^2} = \frac{v \frac{\partial}{\partial f}\left(v \frac{\partial u}{\partial f} - u \frac{\partial v}{\partial f} \right) - \left(v \frac{\partial u}{\partial f} - u \frac{\partial v}{\partial f} \right) 2 \frac{\partial v}{\partial f}}{v^3} = \frac{v^2 \frac{\partial^2 u}{\partial f^2} - uv \frac{\partial^2 v}{\partial f^2} - 2v \frac{\partial u}{\partial f} \frac{\partial v}{\partial f} + 2u \left(\frac{\partial v}{\partial f} \right)^2}{v^3} \tag{10}$$

But we know that

$$u = |<\mathbf{r}_1, \mathbf{W}_2>|^2, \; v = \|\mathbf{W}_2\|^2 - \|\mathbf{W}_1\|^{-2} |<\mathbf{W}_1, \mathbf{W}_2>|^2 \tag{11}$$

where $\mathbf{r}_1$ and $\mathbf{W}_1$ are fixed. Derivatives of u can be calculated directly using [1]. To calculate the derivatives of v, we write

$$v = \|\mathbf{W}_2\|^2 - \|\mathbf{W}_1\|^{-2} w, \; w = |<\mathbf{W}_1, \mathbf{W}_2>|^2 \tag{12}$$

Obviously the derivatives of w can be calculated directly using [1]. Then we may proceed with

$$\frac{\partial v}{\partial f} = -\|\mathbf{W}_1\|^{-2} \frac{\partial w}{\partial f}, \quad \frac{\partial^2 v}{\partial f^2} = -\|\mathbf{W}_1\|^{-2} \frac{\partial^2 w}{\partial f^2} \tag{13}$$

under the adequate assumption that $\|\mathbf{W}_2\|^2$ remain constant regarding variable f.

If $\mathbf{W}_1$ and $\mathbf{W}_2$ are both spectra of unit amplitude zero-central-phase windowed sinusoids using the same window function from the cosine family, then w can be directly calculated from the two frequencies. To show this we write

$$\begin{aligned} <\mathbf{W}_1, \mathbf{W}_2> &= \sum_k W_{1k} W_{2k}^* = \sum_k \sum_n w_n e^{j2\pi f_1 n/N} e^{-j\pi f_1} e^{-j2\pi kn/N} \sum_m w_m e^{-j2\pi f_2 m/N} e^{j\pi f_2} e^{j2\pi km/N} \\ &= e^{-j\pi(f_1-f_2)} \sum_n \sum_m w_n w_m e^{j2\pi(f_1 n - f_2 m)/N} \sum_k e^{-j2\pi k(n-m)/N} = N e^{-j\pi(f_1-f_2)} \sum_n w_n^2 e^{j2\pi(f_1-f_2)n/N} \end{aligned} \tag{14}$$

Notice that (14) is no other than the DC component of a zero-central-phase complex sinusoid of frequency f_1-f_2 bins under window w^2. Now w show that if w is from the cosine window family, then so does w^2. Let w be

$$w_n = \sum_{m=-M}^{M} (-1)^m c_m e^{j2\pi mn/N} \tag{15}$$

then we have

$$w_n^2 = \sum_{l=-M}^{M} (-1)^l c_l e^{j2\pi ln/N} \sum_{k=-M}^{M} (-1)^k c_k e^{j2\pi kn/N} = \sum_{l=-M}^{M} \sum_{k=-M}^{M} (-1)^{l+k} c_l c_k e^{j2\pi(l+k)n/N}. \tag{16}$$

Substituting m for $l+k$ we get

$$w_n^2 = \sum_{m=-2M}^{2M} \sum_{k=-M}^{M} (-1)^m c_{m-k} c_k e^{j2\pi mn/N} = \sum_{m=-2M}^{2M} (-1)^m d_m e^{j2\pi mn/N} \tag{17}$$

where $\mathbf{d}$ defined by

$$d_m = \sum_{k=-M}^{M} c_{m-k} c_k \tag{18}$$

is the convolution of $\mathbf{c}$ with itself. For example, the Hann window has $\mathbf{c}$=(1, 2, 1)/4 and $\mathbf{d}$=(1, 4, 6, 2, 1)/16. Using (17) and Eq. 3 of [1] we get

$$<\mathbf{W}_1, \mathbf{W}_2> = N \sum_{m=-2M}^{2M} d_m S_a^N (f_1 - f_2 + m) e^{-j\pi(f_1-f_2+m)/N} \tag{19}$$

Calculation of w and its derivatives following the thread below:

$$<>_r = N \sum_{m=-2M}^{2M} d_m S_{f_{1,2},m} \cos\omega_{f_{1,2},m}, \quad <>_i = -N \sum_{m=-2M}^{2M} d_m \sin\Omega_{f_{1,2},m} \tag{20}$$

$$\frac{\partial <>_r}{\partial f_1} = N \sum_{m=-2M}^{2M} d_m \left(\dot{S}_{f_{1,2},m} \cos\omega_{f_{1,2},m} - \frac{\pi}{N} \sin\Omega_{f_{1,2},m} \right), \quad \frac{\partial <>_i}{\partial f_1} = -N \sum_{m=-2M}^{2M} d_m \pi \cos\Omega_{f_{1,2},m} \tag{21}$$

$$\frac{\partial^2 <>_r}{\partial f_1^2} = N \sum_{m=-2M}^{2M} d_m \left(\ddot{S}_{f_{1,2},m} \cos\omega_{f_{1,2},m} - \frac{\pi}{N} \dot{S}_{f_{1,2},m} \sin\omega_{f_{1,2},m} - \frac{\pi^2}{N} \cos\Omega_{f_{1,2},m} \right), \quad \frac{\partial^2 <>_i}{\partial f_1^2} = N \sum_{m=-2M}^{2M} d_m \pi^2 \sin\Omega_{f_{1,2},m} \tag{22}$$

$$w = <>_r^2 + <>_i^2, \quad \frac{\partial w}{\partial f_1} = 2 <>_r \frac{\partial <>_r}{\partial f_1} + 2 <>_i \frac{\partial <>_i}{\partial f_1},$$

$$\frac{\partial^2 w}{\partial f_1^2} = 2 <>_r \frac{\partial^2 <>_r}{\partial f_1^2} + 2\left(\frac{\partial <>_r}{\partial f_1}\right)^2 + 2 <>_i \frac{\partial^2 <>_i}{\partial f_1^2} + 2\left(\frac{\partial <>_i}{\partial f_1}\right)^2 \tag{23}$$

where $S_{f_{1,2},m} = S_a^N (f_1 - f_2 + m)$, $\dot{S}_{f_{1,2},m} = (\partial / \partial f_1) S_a^N (f_1 - f_2 + m)$, $\ddot{S}_{f_{1,2},m} = (\partial^2 / \partial f_1^2) S_a^N (f_1 - f_2 + m)$, $\Omega_{f_{1,2},m} = \pi(f_1 - f_2 + m)$, $\omega_{f_{1,2},m} = \Omega_{f_{1,2},m} / N$. Derivatives regarding f_2 are easily given as

$$\frac{\partial w}{\partial f_2} = -\frac{\partial w}{\partial f_1}, \quad \frac{\partial^2 w}{\partial f_2^2} = \frac{\partial^2 w}{\partial f_1^2}. \tag{24}$$

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Reference

[1] Calculation of discrete spectrum of cosine-family-windowed sinusoids.doc.

***Symmetry**

We can show that the decomposition onto $\mathbf{W_1}$ and $\mathbf{W_2}$ is symmetrical in the sense that

$$\lambda_1 = \frac{<\mathbf{X},\mathbf{W}_1>\|\mathbf{W}_2\|^2 - <\mathbf{X},\mathbf{W}_2><\mathbf{W}_2,\mathbf{W}_1>}{\|\mathbf{W}_1\|^2\|\mathbf{W}_2\|^2 - \left|\langle\mathbf{W}_1,\mathbf{W}_2\rangle\right|^2},\ \lambda_2 = \frac{<\mathbf{X},\mathbf{W}_2>\|\mathbf{W}_1\|^2 - <\mathbf{X},\mathbf{W}_1><\mathbf{W}_1,\mathbf{W}_2>}{\|\mathbf{W}_1\|^2\|\mathbf{W}_2\|^2 - \left|\langle\mathbf{W}_1,\mathbf{W}_2\rangle\right|^2}. \tag{25}$$

and that

$$\begin{aligned}
E &= |\mu_1|^2\|\mathbf{W}_1\|^2 + |\mu_2|^2\|\mathbf{V}_2\|^2 = \frac{\left|\langle\mathbf{X},\mathbf{W}_1\rangle\right|^2}{\|\mathbf{W}_1\|^2} + \frac{\left|\langle\mathbf{X},\mathbf{W}_2\rangle\right|^2}{\|\mathbf{W}_2\|^2} \\
&- \frac{\langle\mathbf{W}_1,\mathbf{W}_2\rangle\langle\mathbf{W}_2,\mathbf{X}\rangle\langle\mathbf{X},\mathbf{W}_1\rangle + \langle\mathbf{W}_2,\mathbf{W}_1\rangle\langle\mathbf{W}_1,\mathbf{X}\rangle\langle\mathbf{X},\mathbf{W}_2\rangle}{\|\mathbf{W}_1\|^2\|\mathbf{W}_2\|^2 - \left|\langle\mathbf{W}_1,\mathbf{W}_2\rangle\right|^2} + \frac{\left|\langle\mathbf{W}_1,\mathbf{W}_2\rangle\right|^2}{\|\mathbf{W}_1\|^2\|\mathbf{W}_2\|^2}\frac{\left|\langle\mathbf{X},\mathbf{W}_1\rangle\right|^2\|\mathbf{W}_2\|^2 + \left|\langle\mathbf{X},\mathbf{W}_2\rangle\right|^2\|\mathbf{W}_1\|^2}{\|\mathbf{W}_1\|^2\|\mathbf{W}_2\|^2 - \left|\langle\mathbf{W}_1,\mathbf{W}_2\rangle\right|^2} \\
&= \frac{\left|\langle\mathbf{X},\mathbf{W}_1\rangle\right|^2\|\mathbf{W}_2\|^2 + \left|\langle\mathbf{X},\mathbf{W}_2\rangle\right|^2\|\mathbf{W}_1\|^2 - \langle\mathbf{W}_1,\mathbf{W}_2\rangle\langle\mathbf{W}_2,\mathbf{X}\rangle\langle\mathbf{X},\mathbf{W}_1\rangle - \langle\mathbf{W}_2,\mathbf{W}_1\rangle\langle\mathbf{W}_1,\mathbf{X}\rangle\langle\mathbf{X},\mathbf{W}_2\rangle}{\|\mathbf{W}_1\|^2\|\mathbf{W}_2\|^2 - \left|\langle\mathbf{W}_1,\mathbf{W}_2\rangle\right|^2}
\end{aligned} \tag{26}$$

However, these are too much for calculating the values with.