Calculation of biorthogonal spline wavelet filters
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The spline wavelets are calculated using perfect reconstruction filters (h, g) and ($\tilde{h}$, $\tilde{g}$), so that

$\hat{g}(\omega) = e^{-i\omega}\hat{\tilde{h}}^*(\omega+\pi)$, $\hat{\tilde{g}}(\omega) = e^{-i\omega}\hat{h}^*(\omega+\pi)$,

$\hat{h}(\omega) = \sqrt{2}\exp\left(\frac{-i\varepsilon\omega}{2}\right)\left(\cos\frac{\omega}{2}\right)^p L(\cos\omega)$,

$\hat{\tilde{h}}(\omega) = \sqrt{2}\exp\left(\frac{-i\varepsilon\omega}{2}\right)\left(\cos\frac{\omega}{2}\right)^{\tilde{p}} \tilde{L}(\cos\omega)$,

where p and $\tilde{p}$ are the same parity, ε=1 for p even and 1 for p odd, and

$L(\cos\omega)\tilde{L}(\cos\omega) = P\left(\sin^2\frac{\omega}{2}\right)$, $P(x) = \sum_{k=0}^{q-1} C_{q-1+k}^k y^k$, $q = (p+\tilde{p})/2$,

so that the perfect reconstruction condition $\hat{h}^*(\omega)\hat{\tilde{h}}(\omega) + \hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega+\pi) = 2$ is met.

Let L(cosω)=1, $\tilde{L}(\cos\omega) = P\left(\sin^2\frac{\omega}{2}\right)$.

Then $\hat{h}(\omega) = \sqrt{2}\exp\left(\frac{-i\varepsilon\omega}{2}\right)\left(\cos\frac{\omega}{2}\right)^p$, $\hat{\tilde{h}}(\omega) = \sqrt{2}\exp\left(\frac{-i\varepsilon\omega}{2}\right)\left(\cos\frac{\omega}{2}\right)^{\tilde{p}} \sum_{k=0}^{q-1} C_{q-1+k}^k\left(\sin^2\frac{\omega}{2}\right)^k$.

The length of h is p+1, of $\tilde{h}$ is p+2 $\tilde{p}$ -1. When p is even, h and $\tilde{h}$ are symmetric regarding 0.

$$\hat{h}(\omega) = \sqrt{2}\left(\cos\frac{\omega}{2}\right)^p = \sqrt{2}\left(\frac{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}{2}\right)^p = \sqrt{2}\,\frac{\sum_{k=0}^{p} C_p^k e^{jk\frac{\omega}{2}} e^{-j(p-k)\frac{\omega}{2}}}{2^p} = \sqrt{2}\sum_{n=-p/2}^{p/2}\frac{C_p^{n+p/2}}{2^p}e^{-j\omega n} ,$$

i.e. $h(n) = \sqrt{2}\,\frac{C_p^{n+p/2}}{2^p},\ n = -\frac{p}{2}, -\frac{p}{2}+1, \cdots, \frac{p}{2}-1, \frac{p}{2}$.

To calculate $\tilde{h}$, first we calculate an h_1 corresponding to $\sqrt{2}\exp\left(\cos\frac{\omega}{2}\right)^{\tilde{p}}$, which is simply

$h_1(n) = \sqrt{2}\,\frac{C_{\tilde{p}}^{n+\tilde{p}/2}}{2^{\tilde{p}}}$. Then we calculate an h_2 that corresponds to $\sum_{k=0}^{q-1} C_{q-1+k}^k\left(\sin^2\frac{\omega}{2}\right)^k$, that is

$$h_2(\omega) = \sum_{k=0}^{q-1} C_{q-1+k}^k\left(\sin^2\frac{\omega}{2}\right)^k = \sum_{k=0}^{q-1} C_{q-1+k}^k\left(\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{j2}\right)^{2k} = \sum_{k=0}^{q-1} C_{q-1+k}^k\,\frac{\sum_{l=0}^{2k} C_{2k}^l e^{j\frac{\omega}{2}l} e^{-j\frac{\omega}{2}(2k-l)}(-1)^l}{(-1)^k 2^{2k}}$$

$$= \sum_{k=0}^{q-1}\sum_{l=0}^{2k}\frac{C_{q-1+k}^k C_{2k}^l}{2^{2k}}e^{-j\omega(k-l)}(-1)^{k-l} \overset{n=k-l}{=} \sum_{k=0}^{q-1}\sum_{n=-k}^{k}\frac{C_{q-1+k}^k C_{2k}^{k-n}}{2^{2k}}(-1)^n e^{-j\omega n} = \sum_{n=1-q}^{q-1}(-1)^n\sum_{k=|n|}^{q-1}\frac{C_{q-1+k}^k C_{2k}^{k-n}}{2^{2k}}e^{-j\omega n}$$

so that

$h_2(n) = (-1)^n\sum_{k=|n|}^{q-1}\frac{C_{q-1+k}^k C_{2k}^{k-n}}{2^{2k}},\quad n = 1-q, 2-q, \cdots, q-1$. $\tilde{h}$ is then calculated as h_1*h_2.

When p is odd, h and $\tilde{h}$ are symmetric regarding 0.5, and

$$\hat{h}(\omega) = \sqrt{2}e^{-\frac{j\omega}{2}}\left(\cos\frac{\omega}{2}\right)^p == \sqrt{2}e^{-\frac{j\omega}{2}}\,\frac{\sum_{k=0}^{p} C_p^k e^{jk\frac{\omega}{2}} e^{-j(p-k)\frac{\omega}{2}}}{2^p} = \sqrt{2}\sum_{n=(1-p)/2}^{(p+1)/2}\frac{C_p^{n+(p-1)/2}}{2^p}e^{-j\omega n}$$

So that $h(n) = \sqrt{2}\,\frac{C_p^{n+\frac{p-1}{2}}}{2^p},\ n = -\frac{p-1}{2}, -\frac{p-1}{2}+1, \cdots, \frac{p-1}{2}, \frac{p+1}{2}$. Similarly $h_1(n) = \sqrt{2}\,\frac{C_{\tilde{p}}^{n+\frac{\tilde{p}-1}{2}}}{2^{\tilde{p}}}$, h_2(n) is the same thing as the p even case, i.e. $h_2(n) = (-1)^n\sum_{k=|n|}^{q-1}\frac{C_{q-1+k}^k C_{2k}^{k-n}}{2^{2k}},\quad n = 1-q, 2-q, \cdots, q-1$.

Then $\tilde{h}$ is then calculated as h_1*h_2. In the end $g(n)=(-1)^{1-n}\tilde{h}(1-n)$, $\tilde{g}(n)=(-1)^{1-n}h(1-n)$.

***Practical issues on deriving the filter pairs**

All the four filters have the same length parity, i.e. all even or all odd. Odd parity corresponds to even values of p and $\tilde{p}$, with the filters h and $\tilde{h}$ being symmetric regarding 0, so that neither the forward transform nor the backward reconstruction introduces a time shift. g and $\tilde{g}$ are obtained by multiplying $\tilde{h}$ and h with alternate sign sequence …-1, 1, -1, 1, …, so that the symmetric centres h(0) and $\tilde{h}(0)$ are always multiplied with +1. However, there must be a ±1 time shift to transfer $(-1)^n$h(n) to $\tilde{g}(n)$, or $(-1)^n\tilde{h}(n)$ to g(n). To keep a zero time shift, we make one -1 and the other +1 (actually nothing stops us using ±3, or ±5, etc., but never use something like ±2, ±4) . So that one of g and $\tilde{g}$ is symmetric regarding 1, the other is symmetric regarding -1. This is summarized as follows, where l is the length of h and $\tilde{g}$, hl=(l-1)/2, L is the length of $\tilde{h}$ and g, hL=(L-1)/2, so that
p and $\tilde{p}$ are even;
l=p+1 is odd;
L=p+2 $\tilde{p}$ -1 is odd;
hl=p/2, hL=p/2+ $\tilde{p}$ -1 is odd samples from hl.

filter	first point	last point	centre	
h	-hl	hl	0	
$\tilde{h}$	-hL	hL	0	
g	-hL+1	hL+1	1	$g(k)=(-1)^{1-k}\tilde{h}(1-k)$
$\tilde{g}$	-hl-1	hl-1	-1	$\tilde{g}(k)=(-1)^{-1-k}h(-1-k)$

On the other hand, even parity corresponds to odd values of p and $\tilde{p}$, with the filters h and $\tilde{h}$ being symmetric regarding -0.5. However, this implies the combination of h and $\tilde{h}$ introduces a time shift of 1. To correct this, we right shift one of them, say $\tilde{h}$, so that it is symmetric regarding 0.5, to make the combined time shift zero. g and $\tilde{g}$ are obtained by multiplying $\tilde{h}(-n)$ and h(-n) with alternate sign sequence …-1, 1, -1, 1, …, so that h(0) and $\tilde{h}(0)$, after the shift operation, are both multiplied by +1. g is anti-symmetric regarding -0.5, $\tilde{g}$ is anti-symmetric regarding 0.5. This is summerized as follows, where l is the length of h and $\tilde{g}$, hl=l/2, L is the length of $\tilde{h}$ and g, hL=L/2, so that
p and $\tilde{p}$ are odd;
l=p+1 is even;
L=p+2 $\tilde{p}$ -1 is even;
hl=(p+1)/2, hL=(p-1)/2+ $\tilde{p}$ is even samples from hl.

filter	first point	last point	centre	
h	-hl	hl-1	-0.5	
$\tilde{h}$	-hL+1	hL	0.5	
g	-hL	hL-1	-0.5	$g(k)=(-1)^{k}\tilde{h}(-k)$
$\tilde{g}$	-hl+1	hl	0.5	$\tilde{g}(k)=(-1)^{k}h(-k)$

We are free to shift h by δh, $\tilde{h}$ by –δh, g by δg and $\tilde{g}$ by –δg to get another shifted set of filters, as long as δh+δg is even. In general, if the total shift applied to h and $\tilde{h}$ equals that applied to g and $\tilde{g}$, and the total shift applied to h and g is even, then the whole system makes a time shifting unit. In particular, if we use all the filters as if starting from 0, the whole system introduces a time shift of p+ $\tilde{p}$ -1, regardless of the parity of p.

To test the correctness of the filters, we see if

$$\begin{cases} h * \tilde{h} + g * \tilde{g} = 2\delta(n) \\ [(-1)^n h] * \tilde{h} + [(-1)^n g] * \tilde{g} = 0 \end{cases}.$$

The conditions can be tested in a way as if all the filters start from zero, thanks to the alignments and parities.